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AN EXTENSION OF A RESULT OF RIBE

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ABSTRACT

It is proved that for every $1 \leqq p < \infty$, $1 \leqq q < \infty$ and for every sequence $\{p_n\}$, $1 \leqq p_n < \infty$, $p_n \to p$, the space $X = (\Sigma \oplus l_{p_n})_q$ (resp. $U = (\Sigma \oplus L_{p_n}(0,1))_q$) is uniformly homeomorphic to $X \oplus l_p$ (resp. $U \oplus L_p(0,1)$). This extends Ribe's result from the case $p = 1$ to general $p < \infty$ and thus provides examples of uniformly convex, uniformly homeomorphic Banach spaces which are not Lipschitz equivalent.

Recently Ribe [3] solved the problem of existence of two separable infinite-dimensional Banach spaces which are uniformly homeomorphic, but not Lipschitz equivalent. His example was $X = (\Sigma \oplus L_{p_n})_q$ and $X \oplus L_p$ where $1 < q < \infty$, $1 < p_n < \infty$, $p_n \to p$ and $p = 1$. As he noted at the end of his paper, his proof does not seem to apply to $p > 1$. Since it is desirable to have examples which are superreflexive, there is some interest in extending Ribe's result to the case $p > 1$. In this note we point out how to modify slightly Ribe's argument so as to apply for all $1 \leqq p < \infty$.

Our main result is the following:

THEOREM 1. *Let* $1 \leqq p < \infty$, $\{p_n\}_{n=1}^{\infty}$ *satisfy* $1 \leqq p_n < \infty$ *and* $\lim p_n = p$. *Let* $1 \leqq q < \infty$, *and let* $X = (\Sigma \oplus l_{p_n})_q$ *(resp.* $U = (\Sigma \oplus L_{p_n}(0,1))_q$*). Then* X *(resp.* U*) is uniformly homeomorphic to* $Y = X \oplus l_p$ *(resp.* $V = U \oplus L_p(0,1)$*).*

It is clear that for $p_n \neq p$, $q \neq p$, X does not contain any subspace isomorphic to l_p. Hence, cf. [1], Y is not Lipschitz homeomorphic to any subset of X.

To prove the theorem we need some lemmas.

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LEMMA 2. *For every positive real number* α, *there exists a* $\beta = \beta(\alpha) > 0$ *such that for every Banach space* X, *and for every linear operator* T *from* X *onto* X *satisfying for every* $x \in X$

$$\frac{1}{\alpha}\|x\| \leqq \|Tx\| \leqq \alpha\|x\|,$$

the function $S: X \to X$ *defined by*

$$Sx = \begin{cases} \dfrac{\|x\| \cdot Tx}{\|Tx\|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is a Lipschitz homeomorphism of X *onto itself satisfying for every* $x, y \in X$

$$\frac{1}{\beta}\|x-y\| \leqq \|Sx - Sy\| \leqq \beta\|x-y\|.$$

PROOF. By direct verification it is checked that $\beta = 2\alpha^2 + 1$ works. □

As a consequence we get:

LEMMA 3. *Let* X *be a Banach space for which* GL(X) (= *the group of isomorphisms*) *is connected. Then for every invertible isometry* T *of* X *onto itself, there exist maps* $\{F_t \mid 0 \leqq t \leqq 1\}$ *from* X *onto itself satisfying*:

(a) $F_0 = I$, $F_1 = T$;

(b) $\{F_t \mid 0 \leqq t \leqq 1\}$ *are equi-Lipschitz homeomorphisms* (*i.e.* $\exists K > 0$ *such that for every* $x, y \in X$ *and every* $0 \leqq t \leqq 1$, $(1/K)\|x-y\| \leqq \|F_t x - F_t y\| \leqq K\|x-y\|$);

(c) *for every* $0 \leqq t \leqq 1$ *and for every* $x \in X$, $\|F_t x\| = \|x\|$;

(d) *there exists a* $K > 0$ *such that for every* $x \in X$ *and for every* $0 \leqq t, s \leqq 1$,

$$\|F_t x - F_s x\| \leqq K|t-s|\,\|x\|,$$

$$\|F_t^{-1}x - F_s^{-1}x\| \leqq K|t-s|\,\|x\|.$$

PROOF. Since GL(X) is arcwise connected, there exists a continuous map $g:[0,1] \to \mathrm{GL}(X)$ such that $g(0) = I$, $g(1) = T$. Without loss of generality, g satisfies a Lipschitz condition (g can even be taken as a piecewise linear map). It is also clear that the maps $\{g(t) \mid 0 \leqq t \leqq 1\}$ are equi-isomorphisms of X (i.e., there exists $K_1 > 0$ such that for every $0 \leqq t \leqq 1$ and for every $x \in X$, $\|x\|/K_1 \leqq \|g(t)(x)\| \leqq K_1\|x\|$).

Define now, for every $0 \leqq t \leqq 1$,

$$F_t(x)=\begin{cases}\dfrac{\|x\|g(t)(x)}{\|g(t)(x)\|}, & x\neq 0,\\[2ex] 0, & x=0.\end{cases}$$

Note that

$$F_t^{-1}(x)=\begin{cases}\dfrac{\|x\|g(t)^{-1}(x)}{\|g(t)^{-1}(x)\|}, & x\neq 0,\\[2ex] 0, & x=0.\end{cases}$$

Using Lemma 2, it is easy to see that the maps $\{F_t \mid 0\leqq t\leqq 1\}$ satisfy the requirements of the lemma. □

Before stating the next lemma we recall the definition of Mazur's map $M_{p,q}: l_p \to l_q$, $1\leqq p,q<\infty$:

$$M_{p,q}x(i)=\operatorname{sign} x(i)\,|x(i)|^{p/q}, \qquad i=1,2,\dots .$$

It is easy to check, and well known, that for all p,q and x, $\|M_{p,q}x\|^q=\|x\|^p$, and that for every $\gamma>0$ the maps $\{M_{p,q}\mid 1\leqq p,q<\infty\}$ are equi-uniformly continuous on $\{x;\|x\|\leqq \gamma e^{1/|p-q|}\}$ (see e.g. [3] lemma 1).

We are going to use the following notation: for two sequences $x=(x_1,x_2,x_3,\dots)$ and $y=(y_1,y_2,\dots)$ of real numbers we let $\langle x,y\rangle$ be the sequence $(x_1,y_1,x_2,y_2,\dots)$. In the direct sum $l_p\oplus l_q\oplus l_r$ we take as the norm of an element the sum of the norms of its three components.

LEMMA 4. *Let* $1\leqq p<\infty$. *Then for every* $1\leqq q,r<\infty$ *there exist maps*

$$\{G_t=G_{t,q,r}: l_p\oplus l_q\oplus l_r\to l_q\oplus l_r \mid 0\leqq t\leqq 1\}$$

such that:

(a) *For every* $0\leqq t\leqq 1$, G_t *is a one to one map onto* $l_q\oplus l_r$.

(b) *For every* $\gamma>1$ *the maps*

$$\{G_{t,q,r}\mid 0\leqq t\leqq 1, 1\leqq q,r<\gamma\}$$

are equi-uniformly continuous on the sets

$$D_{q,r}=\{w\in l_p\oplus l_q\oplus l_r \mid \|w\|\leqq \gamma e^{1/c_{q,r}}\}$$

where $c_{q,r}=\max\{|p-q|,|p-r|\}$, *and the maps*

$$\{G_{t,q,r}^{-1}\mid 0\leqq t\leqq 1, 1\leqq q,r<\gamma\}$$

are equi-uniformly continuous on the sets $G_{t,q,r}(D_{q,r})$.

(c) *For every* $\gamma > 1$, *the maps* $G_{t/b,q,r}(w)$ *and* $G^{-1}_{t/b,q,r}(w)$ *with* $1 \leqq q, r < \gamma$, $\|w\| \leqq b$ *and* $1 \leqq b \leqq \gamma e^{1/c_{q,r}}$ *are equi-uniformly continuous functions of* t *on* $[0, b]$.

(d) *For every* $0 \leqq t \leqq 1$, *let* $G_t(z, x, y) = (u_t, v_t)$, *then*

$$\|z\|^p + \|x\|^q + \|y\|^r = \|u_t\|^q + \|v_t\|^p.$$

(e) $G_0(z, x, y) = (\langle M_{p,q}z, x\rangle, y)$, $G_1(z, x, y) = (x, \langle M_{p,r}z, y\rangle)$.

PROOF. For a sequence $x = (x_1, x_2, x_3, \dots)$ of reals put $x^O = (x_1, x_3, x_5, \dots)$ and $x^E = (x_2, x_4, x_6, \dots)$. Observe that in our notation $x = \langle x^O, x^E\rangle$.

Define $T : l_p \oplus l_p \to l_p \oplus l_p$ by

$$T(x, y) = (x^E, \langle x^O, y\rangle).$$

By identifying $l_p \oplus l_p$ with l_p we can consider T as an isometry of l_p onto itself. By [2], $\mathrm{GL}(l_p)$ is arcwise connected. By Lemma 3, there are maps $\{F_t \mid 0 \leqq t \leqq 1\}$ which satisfy requirements (a) through (d) of that lemma.

Let $M : l_q \oplus l_r \to l_p \oplus l_p$ be defined by

$$M(x, y) = (M_{q,p}(x), M_{r,p}(y)),$$

and put

$$G_t = M^{-1}F_tMG_0, \qquad 0 \leqq t \leqq 1.$$

It is easy to see that $\{G_t \mid 0 \leqq t \leqq 1\}$ satisfy the requirements of the lemma. □

Of course, the lemma holds (with only minor changes in notation) also in the case of L_p spaces. With Lemma 4 at our disposal we can now follow Ribe's argument to complete the proof of Theorem 1. What follows is a reformulation of the argument in Ribe's paper [3].

PROOF OF THEOREM 1. Without loss of generality we can assume that for every n

$$\left|\frac{p_n}{p} - 1\right| < \frac{\alpha}{pn}$$

where $0 < \alpha < 1/100$.

For every $y \in l_p \oplus X$, $y = (x_0, x_1, x_2, \dots)$ define

$$u_n(y) = \left((\|x_0\|^p + \|x_n\|^{p_n} + \|x_{n+1}\|^{p_{n+1}})^{q/p} + \sum_{i \neq 0, n, n+1} \|x_i\|^q\right)^{1/q}.$$

We consider X in an obvious way as a subspace of $l_p \oplus X$, and thus functions like

u_n are also defined on X. Let $d = 10^{10}$ and let $b = 10d^{\alpha}$. Then $1 < b < d^{1/8}$, and for every $x, y \in l_p \oplus X$ satisfying

$$\|x\|, \|y\| \leqq d^n \quad \text{or} \quad u_n(x), u_n(y) \leqq d^n$$

we have

$$\frac{1}{b}\|x\| \leqq u_n(x) \leqq b\|x\|$$

and

$$|u_n(x) - u_n(y)| \leqq b\|x - y\|.$$

Denote now for every $n \geqq 1$:

$$A_n = \{y \in l_p \oplus X : d^{n-1} \leqq \|y\| \leqq d^n\},$$

$$B_n = \{y \in l_p \oplus X : b^3 d^{n-1} \leqq u_n(y) \leqq d^n / b^3\},$$

$$B_n^- = \{y \in l_p \oplus X : u_n(y) \leqq b^3 d^{n-1}\},$$

$$B_n^+ = \{y \in l_p \oplus X : u_n(y) \geqq d^n / b^3\},$$

$$C_n = A_n \cup (B_{n-1}^+ \cap A_{n-1}) \cup (B_{n+1}^- \cap A_{n+1}) \qquad \text{(where } B_0^+ \text{ is the empty set).}$$

Define now for every $y \in l_p \oplus X$

$$t_n(y) = \begin{cases} 0, & y \in B_n^-, \\ \dfrac{u_n(y) - b^3 d^{n-1}}{d^n / b^3 - b^3 d^{n-1}}, & y \in B_n, \\ 1, & y \in B_n^+. \end{cases}$$

Clearly $0 \leqq t_n(y) \leqq 1$ for all n and y.

For every n, let P_n be the projection of $l_p \oplus X$ on $l_p \oplus l_{p_n} \oplus l_{p_{n+1}}$, and let $Q_n = I - P_n$.

We define now $h : l_p \oplus X \to X$. If $y \in A_n$, $n \geqq 2$ we define

$$h(y) = G_{t_n(y), p_n, p_{n+1}}(P_n(y)) \oplus Q_n(y)$$

and for $\|y\| \leqq d$, we define

$$h(y) = G_{1, p_1, p_2}(P_2(y)) \oplus Q_2(y).$$

It is easy to see that for every n and for every $y \in B_n^+ \cap B_{n+1}^-$ we have

$$h(y) = G_{1, p_n, p_{n+1}}(P_n(y)) \oplus Q_n(y) = G_{0, p_{n+1}, p_{n+2}}(P_{n+1}(y)) \oplus Q_{n+1}(y).$$

Since $\{x : \|x\| = d^n\} \subset B_n^+ \cap B_{n+1}^-$, it follows that the definition of h is consistent.

We shall prove now that h is a uniform homeomorphism. Notice first that for every $y \in C_n$, $u_n(h(y)) = u_n(y)$. Hence $h(B_n) \subset B_n \cap X$ for every n.

It is also clear that for every $n \geqq 1$

$$h((B_n^+ \cap A_n) \cup (B_{n+1}^- \cap A_{n+1})) \subset [(B_n^+ \cap A_n) \cup (B_{n+1}^- \cap A_{n+1})] \cap X.$$

Hence $h(C_n) \subset C_n \cap X$ for every n.

Now for every $y_1, y_2 \in l_p \oplus X$ such that $\|y_1 - y_2\| \leqq 1$, there exists an $n \geqq 1$ such that $y_1, y_2 \in C_n$. Hence for $i = 1, 2$

$$h(y_i) = G_{t_n(y_i), p_n, p_{n+1}}(P_n(y_i)) \oplus Q_n(y_i).$$

The uniform continuity of h follows now by (b) and (c) of Lemma 4.

Notice that for every $x \in C_n \cap X$, $h^{-1}(x) = G^{-1}_{t_n(x), p_n, p_{n+1}}(P_n(x)) \oplus Q_n(x)$. Hence, by the same reasoning, h^{-1} is uniformly continuous. This proves the theorem. □

References

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